Algebra, Topology and Geometry of Groups G_n^k

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Outline

The main principle

If dynamical systems describing a motion of n particles possess a nice codimension 1 property governed by k particles then they have topological invariants valued in groups G_n^k.

The definition of G_n^k: Free k-Braids

For two integers $n > k$, we define the group G_n^k as the group having the following $\binom{n}{k}$ generators a_m, where m runs the set of all unordered k-tuples $m_1, \dots, m_k$, whereas each m_i are pairwise distinct numbers from $\{1, \dots, n\}$.

$$G_n^k = \langle a_m | (1), (2), (3) \rangle.$$

Free k-Braids:Relations

For each $(k+1)$-tuple U of indices $u_1, \ldots, u_{k+1} \in \{1, \ldots, n\}$,consider the $k+1$ sets $m^j = U \backslash \{u_j\}, j = 1, \ldots, k+1$. With U, we associate the relation

$$a_{m^1} \cdot a_{m^2} \cdots a_{m^{k+1}} = a_{m^{k+1}} \cdots a_{m^2} \cdot a_{m^1}; \tag{1}$$

for two tuples U and $\bar{U}$, which differ by order reversal, we get the same relation.

Thus, we totally have $\frac{(k+1)!\binom{n}{(k+1)}}{2}$ relations.

We shall call them the *tetrahedron relations*.

Free k-Braids:Relations

For k-tuples m, m' with $Card(m \cap m') < k - 1$, consider the *far commutativity relation:*

$$a_m a_{m'} = a_{m'} a_m \qquad (2).$$

Note that the far commutativity relation can occur only if $n > k + 1$. Besides that, for all multiindices m, we write down the following relation:

$$a_m^2 = 1 \qquad (3)$$

Define G_n^k as the quotient group of the free group generated by all a_m for all multiindices m by relations (1), (2) and (3).

Groups G_n^k have many nice mappings

The strand deleting homomorphisms: $G_n^k \to G_{n-1}^k$
Take all $a_{ijk} \to 1$ if one of i, j, k is equal to n; otherwise, take a_{ijk} to a_{ijk}.
The strand forgetting homomorphism: $G_n^k \to G_{n-1}^{k-1}$
We take $a_{ijn} \to a_{ij}$; $a_{pqr} \to 1$ if $p \neq n, q \neq n, r \neq n$.

$$C_n^k = C_{n-1}^k + C_{n-1}^{k-1}$$

(Russian):

$$\binom{n}{k} = \binom{(n-1)}{k} + \binom{(n-1)}{(k-1)}.$$

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Dynamical System of points in $\mathbb{R}^2$

Where do G_n^k come from?

Consider a collection of pairwise distinct point in $\mathbb{R}^2 = \mathbb{C}^1$. A motion of such points will give rise to a dynamical system. .

Say that points are in general position if no 3 of them are on the same line.

We shall mark those moments when they are not in general position: if some three points z_i, z_j, z_k are collinear, we write down a letter a_{ijk}.

With this moment we associate a_{ijk}; with the braid we associate the product of all critical values a_{ijk}.

We shall need some non-degeneracy conditions on critical values.

There are recent modifications of the group G_n^3, when the order of points is taken into account: a'_{ijk}, a'_{jik}, a'_{jki} are different generators.

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The pure braid group

More precisely, let $n \in \mathbb{N}$. Consider the collection of n points $z_j, j = 1, \dots, n, z_j^* = exp(\frac{2\pi ij}{n}) \in \mathbb{C}^1 = \mathbb{R}^2$.
This collection will be thought of as the reference point z^* in the configuration space of ordered n-tuples of different points on the plane.
$PB_n = \pi_1(C_n, z^*)$.

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General Position Deformations: Codimension 1 singularities

In general position, when we perform an isotopy from one braid to another, we can

1. A horizontal (non-degenerate) *quadrisecant*;
2. Two different horizontal trisecants *at the same moment* ;
3. The trajectory of z_j has a *tangency* with the line passing through z_i, z_j.

These three types of singluarities give rise to three types of relations in G_n^3.

Calculations are easy (joint work with I.M.Nikonov)

We get a map from the braid group PB_n to the group G_n^3, see Fig.1.

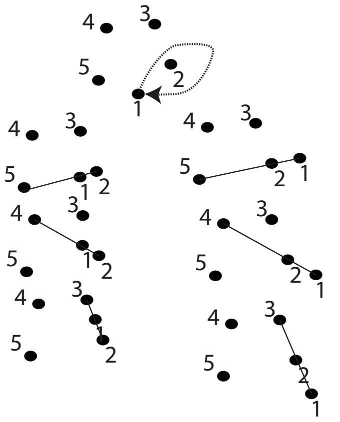

Figure: $a_{125}a_{124}a_{123}a_{125}a_{124}a_{123}$

What else? A map $PB_n \to G_n^4$ (joint work with Nikonov)

We can consider motions of points on the plane and mark those moments when some four points z_i, z_j, z_k, z_l belong to the same circle (or line), we mark them by a generator a_{ijkl}. Then we get a map from PB_n to G_n^4. [Manturov, Nikonov 2015].

What about...

- The *word* problem for G_n^k...
- The *conjugacy* problem for G_n^k...

Let us start with G_n^2.

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The simplest invariants are:
intersection index, linking coefficient.
Number Conservation Law.

Complicated objects: homotopy groups, (π_1).
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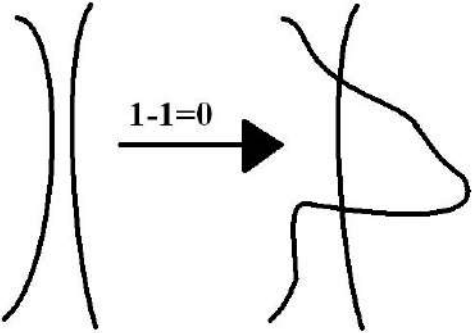

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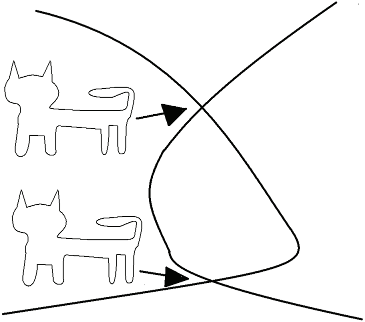
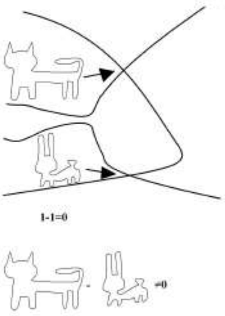


In this picture you may see a Reidemeister move or a holomorphic bigon which cancels two generators of some complex, but I just see two pictures.

Ein fremder Zauberbildersaal entgegen.



Er sähe A und B als Mensch und Tier.

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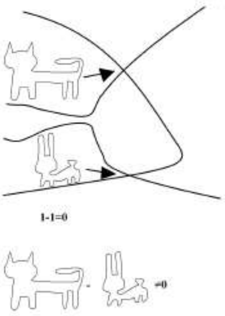

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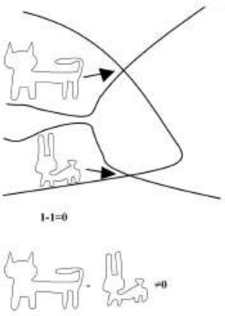

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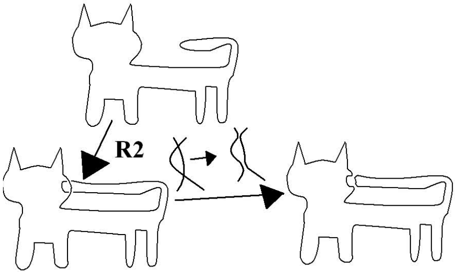
Digression. Virtual Knots. Parity. Cats



If a diagram is complicated enough then it reproduces itself

What is a categorification (Lou Kauffman's interpretation)

Two ways of *cat*egorifying objects:

1. Define an algebraic structure (Hopf, twisted, Frobenius, TQFT, ...) with lots of axioms, and check every time that this axioms are satisfied.
2. Draw cats(for G_n^k-groups: Manturov-Nikonov indices, see ahead).

We want to associate cats with group generators.
Picture-valued invariants provide many new combinatorial complexities.

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Picture-valued invariants provide many new combinatorial complexities.

Word problem and conjugacy problem are solved very easily.
The same principle:
A word

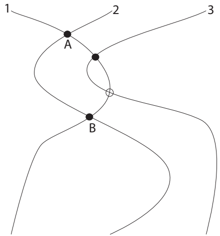
$$abcab^3 \in \mathbb{Z} * \mathbb{Z} * \mathbb{Z} = \langle a, b, c | \rangle$$

is irreducible, thus it appears inside every word equivalent to it.
Exempli gratia:

$$abaa^{-1}cb^{-1}bab^3.$$

Cats (blue letters) can be cancelled only if they are similar.

Pictures for G_n^2-braids (Free, Gaussian)



Both crossings A and B are of type $(1, 2)$; they are both represented by a_{ij} in the group G_3^2, but they are different: the parity consideration coming from the third strand shows that they will never cancel.

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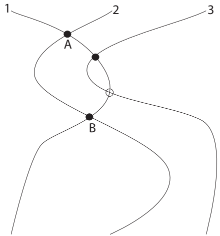

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Free Knots (Knot counterpart of G_n^2)

Free knots (invented by Turaev in 2004 as homotopy classes of Gauss words) are knot theoretic counterpart of G_n^2. Their non-triviality was first shown in [Manturov 2009]; they have lots of powerful picture-valued invariants (cats).

Many nice features of them are realized by the bracket

$$[K] = K,$$

where K on the left hand side is a knot and K on the right hand side is a single diagram of it.

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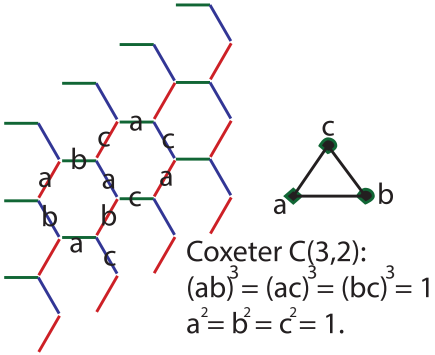

The same Cayley graph gives rise to two different groups G_3^2 and $C(3,2)$: elements of the same colour correspond to the same letters in G_3^2.

G_n^3 and the Coxeter groups

Relations:

$$a_{12}^2 = a_{13}^2 = a_{23}^2 = 1,$$

$$(a_{12}a_{13}a_{23})^2 = 1.$$

A path in the Cayley graph is closed, if every horizontal segment (a_{12}) is cancelled with the same horizontal segment (the same letter with the same abscissa).

This is the geometrical interpretation of indices.

The same rewriting phenomenon (different groups with the same Cayley graph) works for all G_n^2.

These indices, pictures and cats solve problems for G_n^2.

• What about G_n^k with larger k?

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The groups G_n^3 and *imaginary generators*

(joint work with Seongjeong Kim, [Imaginary])

Is the map $PB_n \to G_n^3$ injective? No: spherical braids.

But: By adding some G_n^2 information to G_n^3, we can make this map injective.

Is the map $PB_n \to G_n^3$ surjective? No There are many "non-realizable" braids.

But: In some nice cases (like 4-braids in $\mathbb{R}P^2$, we make all elements of G_4^3 "realizable").

By using braid group techniques, we can **read between letters.**

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An Old Known Example: Artin's generators and enhancement of G_n^2

Dynamical system: pairwise distinct n points on $\mathbb{R}^2$.
Property: two points have the same abscissa (x coordinate).

Artin's Generators are of the G_n^2-nature.

Well, the group we obtain is not quite the same as G_n^2: standard generators of G_n^2 are all involutions, σ_i are not; moreover, the index i of σ_i contains only "local" information on the line and does not tell anything about the numbers endpoints of the braid.

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Now, between the "old" **black** letters σ_i:

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Maybe, like this:

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G_n^k-hierarchy

The G_n^2-property can be thought of as a partial case of the G_{n+1}^3-property: if some two points z_i and z_j belong to the same **vertical** line, we can think that z_i, z_j, and $z_{n+1} = z_\infty$ belong to the same line, where $z_\infty = (0, -\infty)$ is the "infinite point".
After a respective transformation, we can think of all points $z_1, \cdots, z_n$ as lying in the upper half-plane with $z_\infty = z_{n+1} = 0$.

G_n^k-hierarchy

The G_n^3-property can be thought of as a partial case of the G_{n+1}^4-property. Indeed, the equation of a line $bx + cy + d = 0$ can be thought of as a partial case of the equation of a circle $a(x^2 + y^2) + bx + cy + d = 0$ for $a = 0$.

Thus, lines can be thought of as circles passing through the infinite point. From this point of view, we can again think of the generator a_{ijk} of G_n^3 as a generator $a_{ijk\infty}$ of G_{n+1}^4 where $n+1$ means that we have one more "infinite" point. Of course, we can "read" a_{ijkl} between already existing $a_{ijk} = a_{ijk\infty} \dots$

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Word Problem For G_4^3

$$G_4^3 = \langle a, b, c, d | a^2 = b^2 = c^2 = (abcd)^2 = 1 \rangle.$$

Lemma (A.B.Karpov, independently V.O.Manturov)

If an element from G_n^3 can be written in a, b, c, then the irreducible presentation of this type is unique.
In particular, the only word representing the trivial element in G_4^3 is the empty word.

This lemma is proved **geometrically:** A 3-strand braid in $\mathbb{R}P^3$ where three generate the trivial braid, can be **straightened** so that one strand goes around the other points.

Word Problem For G_4^3

For cancellation of adjacent letters d, we use *Manturov-Nikonov invariants* (see below).

Lemma

Two adjacent letters d can be cancelled if and only if $\#a \equiv \#b \equiv \#c \pmod 2$, where $\#a$ denotes the number of occurences of a.

Example. *abcdabcd: d**abc**d; we have $\#a \equiv \#b \equiv \#c \equiv 1 \pmod 2$; so we can "move one d towards another d through a, b, c:"*

*abc**d**abc**d** $\to$ abccba**dd**.*

Conjugacy problem for G_4^3

This problem reduces to the conjugacy problem for braids in $\mathbb{R}P^2$, which, in turn, reduces to the conjugacy problem for the spherical braids.

The Groups G_n^k have many "free invariants"

[Manturov, Nikonov 2015].
For the free group $\mathbb{Z}^{*n}$ (or a free product $\mathbb{Z}_2^{*n}$), one can get easy estimates from below:

For the element

$$abcab^3 \in \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$$

the minimal number of letters is 7 (because the minimal representative $w = abcab^3$ has seven letters.

Points in $\mathbb{R}^3$

When we deal with a dynamics of *n distinct points* on the plane, we assume that **no two points belong to the same** $\mathbb{R}^0 \subset \mathbb{R}^2$, so, we can study the pure braid group PB_n by means of G_n^3. Namely, we can treat the braid group as the fundamental group of the configuration space: $\pi_1(C_n(\mathbb{R}^2))$.

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However, the problem becomes meaningful if we **forbid** triples of collinear points:
$C_n'(\mathbb{R}^3) = \{x_1, \cdots, x_n \in \mathbb{R}^3 |$ no 3 points are on the same line$\}$.
Note that the latter condition includes the condition "no two points coincide".
This allows one to define $C_n'(\mathbb{R}^{k+1})$ by saying that **any** $(k-1)$ **points are in general position.**
Thus, we can study dynamics on the line by G_n^2, dynamics on $\mathbb{R}^2$ by G_n^3, dynamics on $\mathbb{R}^3$ by G_n^4 etc.

Theorem

There are well defined homomorphisms from $\pi_1(C_n'(\mathbb{R}^{k-1})) \to G_n^k$.

Higher G_n^k: points in $\mathbb{R}^3$ [Higher]

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Projective Duality: Lines instead of points

Let us first pass from $\mathbb{R}^{k-1}$ to $\mathbb{R}P^{k-1}$.
We get:

Theorem

There are well defined homomorphisms from $\pi_1(C'_n(\mathbb{R}P^{k-1})) \to G_n^k$.

Now, by using projective duality, we can consider cases of *moving projective hyperplanes* instead of *moving points*.
For example, for $\mathbb{R}P^2$ we have moving lines and for "nice codimension one condition" it suffices to have that any two (lines) intersect at one point and in the neighbourhood of a triple intersection everything is linear.
This allows in fact to consider other spaces of curves.

Can we recognize smooth structures by using G_n^k?

In his first paper on Link Homology, Milnor said that one can study topological spaces by using *link groups* of these spaces.
Having some geometrical "codimension 1 conditions", one can study spaces like $C'_n(M_k)$ and their fundamental groups.
Is it possible to detect any smooth structures on manifolds by using such spaces?

Higher G_n^k: Crucial observation

We can realize all elements of G_{k+1}^k by motion of $k+1$ points in $\mathbb{R}^2$.

Geometric solution to the word problem and the conjugacy problem in G_4^3:

They are just represented by 4-strand braids in $\mathbb{R}P^2$.

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Unsolved problems and work in progress

1. **Algebra** Word Problem, Conjugacy Problem: all except G_n^2, G_4^3. Faithful presentations. Permutohedra.
2. **Geometry** Smooth structures, metrical G_n^k properties. Modifications of G_n^k.
3. **Topology**: Realizability, configuration spaces.
4. **Groups**: Rewriting (which groups beside Coxeter); reading between letters (which groups beside braids)?
5. **Knot Theory**: What are G_n^k-knots? Close up configuration spaces! Kontsevich integral: **Go to Leksin's talk!**
6. **Algebraic Geometry:** G_n^k-hierarchy of equations.

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